Chapter 4: Integration

Definition of Antiderivative

A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I

Representation of Antiderivatives

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of form $G(x) = F(x) + C$, for all x in I where C is constant

Example

$y' = 2$: Find the general solution of the differential equation

1) $y = 2x$ ← antiderivative
2) $y = 2x + C$ ← general solution

Antiderivative Notation

$$\frac{dy}{dx} = f(x)$$

$$dy = f(x)\,dx$$

$$y = \int f(x)\,dx = F(x) + C$$

← constant of integration

integrand

variable of integration

antiderivative of $f(x)$

Basic Integration Rules

Differentiation Formulas

$\frac{d}{dx}[c]=0$

$\frac{d}{dx}[kx]=k$

$\frac{d}{dx}[k f(x)]=k f'(x)$

$\frac{d}{dx}[f(x) \pm g(x)]=f'(x) \pm g'(x)$

$\frac{d}{dx}[x^n]=nx^{-1}$

$\frac{d}{dx}[\sin x]=\cos x$

$\frac{d}{dx}[\cos x]=-\sin x$

$\frac{d}{dx}[\tan x]=\sec^2 x$

$\frac{d}{dx}[\sec x]=\sec x \tan x$

$\frac{d}{dx}[\cot x]=-\csc^2 x$

$\frac{d}{dx}[\csc x]=-\csc x \cot x$

Integration Formulas

$\int 0\, dx = C$

$\int k\, dx = kx + C$

$\int k f(x)\, dx = k \int f(x)\, dx$

$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1$

$\int \cos x\, dx = \sin x + C$

$\int \sin x\, dx = -\cos x + C$

$\int \sec^2 x\, dx = \tan x + C$

$\int \sec x \tan x\, dx = \sec x + C$

$\int \csc^2 x\, dx = -\cot x + C$

$\int \csc x \cot x\, dx = -\csc x + C$

Examples

1) $\int (x+7)dx = \int x\,dx + \int 7\,dx$

$\frac{x^{1+1}}{1+1} + 7x + c$

$= \frac{x^2}{2} + 7x + c$

2) $\int (8x^3 - 9x^2 + 4)dx$

$\frac{8}{4}x^4 - \frac{9}{3}x^3 + 4x + c$

$2x^4 - 3x^3 + 4x + c$

3) $\int \sqrt[3]{x^2}\,dx$

$\int x^{\frac{2}{3}}\,dx$

$\frac{3}{5}x^{\frac{5}{3}} + c$

4 $\int (5\cos x + 4\sin x)dx$

$5\sin x - 4\cos x + c$

Integrating Polynomial Functions

a) $\int dx = \int 1 \, dx$

$= x + c$

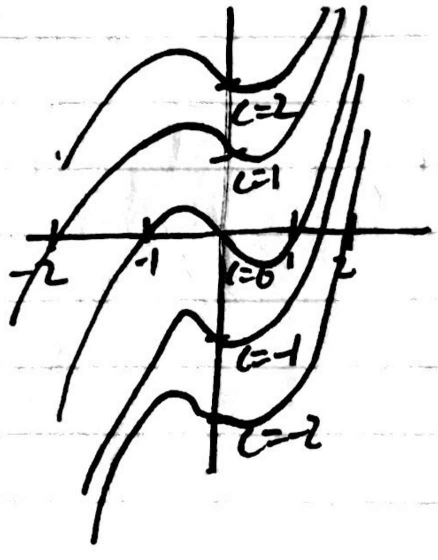
b) $\int (x+2)\, dx = \int x \, dx + \int 2 \, dx$

$= \frac{x^2}{2} + c_1 + 2x + c_2$

$= \frac{x^2}{2} + 2x + C$

Initial Conditions and Particular Solutions

$F(x) = x^3 - x + c$ $\qquad y = \int (3x^2 - 1)\, dx = x^3 - x + C$



Particular Solution

⇒ Find curve that passes through $(2, 4)$

$F(x) = x^3 - x + c$

$F(2) = 4$

$4 = (2^3) - (2) + c$

$c = -2$

$F(x) = x^3 - x - 2$ ⇐ particular solution when passes through $(2, 4)$

Solving a Vertical Motion problem

A ball is thrown upward with an initial velocity of 64 feet per second from an initial height of 80 ft

a) Find the position function given heights as a function of time t

b) When does the ball hit the ground?

Solution

a) $s''(t) = -32$ acceleration due to gravity $= -32$ ft/s

$s'(t) = \int s''(t)\,dt = \int -32\,dt = -32t + C_1$

$s'(0) = 64 = -32(0) + C_1$ using initial velocity

$C_1 = 64$

$s(t) = \int s'(t)\,dt = \int (-32t + 64)\,dt = -16t^2 + 64t + C_2$

$s(0) = 80 = -16(0^2) + 64(0) + C_2$ using initial height

$s(t) = -16t^2 + 64t + 80$

b) $s(t) = -16t^2 + 64t + 80 = 0$

$-16(t+1)(t-5) = 0$

$t = -1, 5$

t is 5 seconds

[4.2] Area

Sigma Notation

The sum of n terms $a_1, a_2, a_3 \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

i is the index of summation

a_i is the ith term of sum

upper and lower bounds of summation are n and 1

Examples of Sigma Notation

a) $\sum_{i=1}^{6} i = 1+2+3+4+5+6$

b) $\sum_{i=0}^{5} (i+1) = 1+2+3+4+5+6$

c) $\sum_{j=3}^{7} j^2 = 3^2+4^2+5^2+6^2+7^2$

d) $\sum_{k=1}^{n} \frac{1}{n}(k^2+1) = \frac{1}{n}(1^2+1) + \frac{1}{n}(2^2+1) \ldots + \frac{1}{n}(n^2+1)$

e) $\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x \ldots + f(x_n)\Delta x$

Properties

1) $\sum_{i=1}^{n} k a_i = k \sum_{i=1}^{n} a_i$

2) $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum b_i$

Summation Formulas	
1) $\sum_{i=1}^{n} c = cn$	2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$	4) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

e.g.

$\sum_{i=1}^{n} \frac{i+1}{n^2}$ for $n = 10, 100, 1000$, and $10,000$

$$\sum_{i=1}^{n} \frac{i+1}{n^2} = \frac{1}{n^2} \sum_{i=1}^{n} i+1$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \right)$$

$$= \frac{1}{n^2}\left[\frac{n(n+1)}{2} + n \right]$$

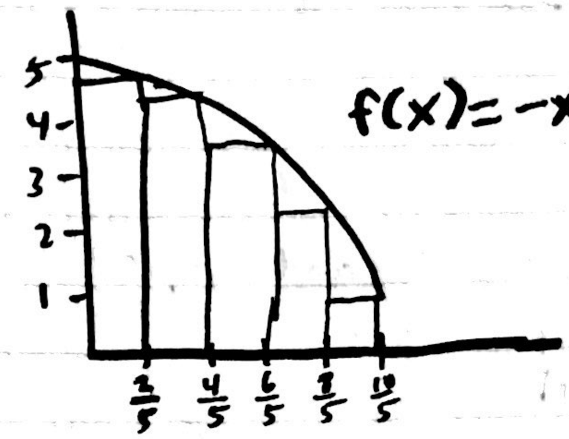
$$= \frac{1}{n^2}\left[\frac{n^2+3n}{2} \right]$$

$$= \frac{n+3}{2n}$$

n	$\sum_{i=1}^{n} \frac{i+1}{n^2} = \frac{n+3}{2n}$
10	0.65000
100	0.51500
1000	0.50150
10000	0.50015

The Area of a Plane Region

Approximating the Area of a Plane Region



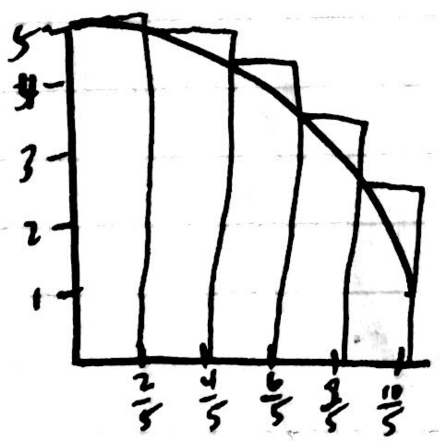
$f(x) = -x^2+5$

The right endpoints of the five intervals are $\frac{2}{5}i$ $i=1,2,3,4,5$

width $=\frac{2}{5}$, height $= f(\frac{2}{5}i)$

$$\sum_{i=1}^{5} f\left(\frac{2i}{5}\right)\left(\frac{2}{5}\right) = \sum_{i=1}^{5}\left[-\left(\frac{2}{5}i\right)^2+5\right]\left(\frac{2}{5}\right) = \frac{162}{25} = 6.48$$

Area > 6.48



$f(x) = -x^2+5$

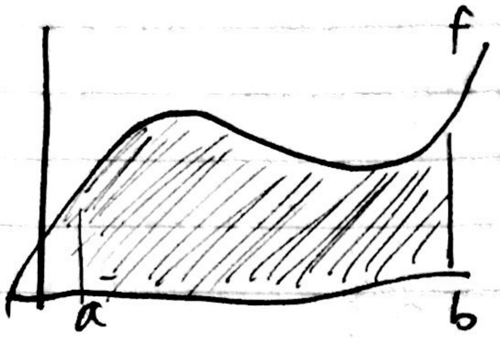
The left endpoints are $\frac{2}{5}(i-1)$ where $i=1,2,3,4,5$

width $=\frac{2}{5}$ height $= f(\frac{2}{5}i-1)$

$$\sum_{i=1}^{5} f\left(\frac{2}{5}i-1\right)\left(\frac{2}{5}\right) = \sum_{i=1}^{5}\left[-\left(\frac{2}{5}i-1\right)^2+5\right]\left(\frac{2}{5}\right) = \frac{202}{25} = 8.08$$

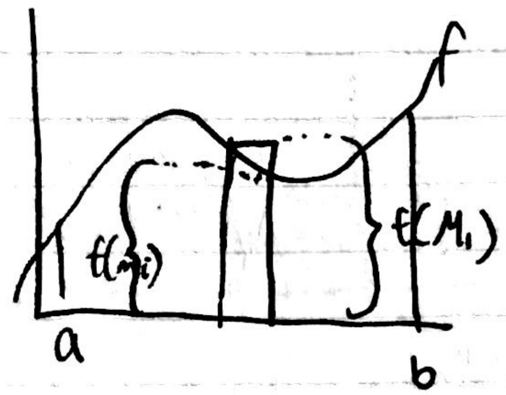
$\boxed{6.48 < \text{Area} < 8.08}$

Upper and Lower Sums



Subdivide $[a,b]$ into n subintervals with width
$$\Delta x = \frac{(b-a)}{n}$$

$f(m_i)$ = Minimum value of $f(x)$
$f(M_i)$ = Maximum value of $f(x)$



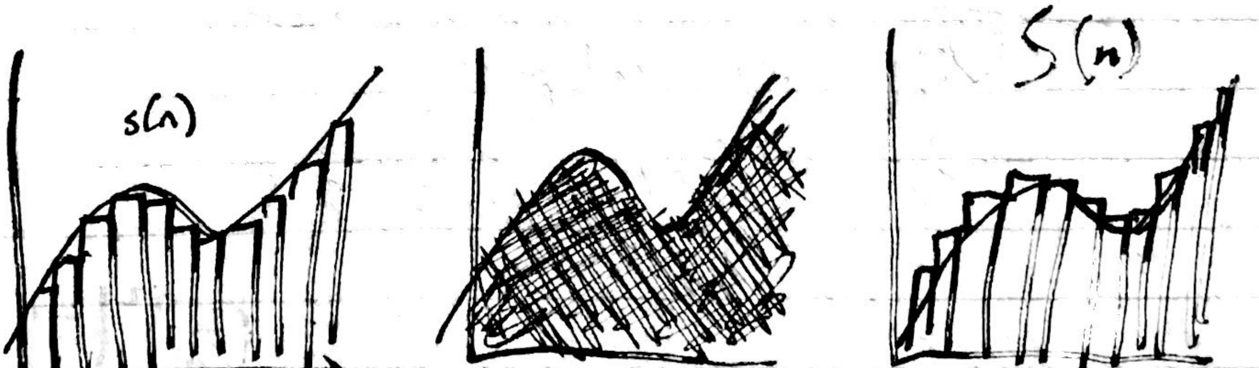
inscribed rectangle - inside ith region
circumscribed rectangle - outside ith region

$$\left(\text{Area of inscribed rectangle}\right) = f(m_i)\Delta x \le f(M_i)\Delta x = \left(\text{Area of circumscribed rectangle}\right)$$

$$\text{Lower sum} = s(n) = \sum_{i=1}^{n} f(m_i)\Delta x \qquad \text{Area of inscribed rectangles}$$

$$\text{Upper Sum} = S(n) = \sum_{i=1}^{n} f(M_i)\Delta x \qquad \text{Area of circumscribed rectangles}$$

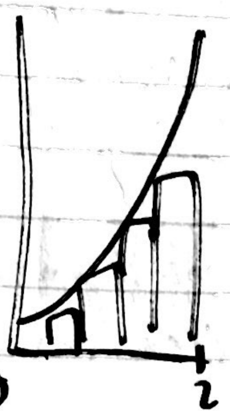
$$s(n) \le (\text{Area of region}) \le S(n)$$



inscribed rectangles < area < circumscribed rectangles

Finding Upper and Lower Sums for a Region

$$\Delta x = \frac{b-a}{n}$$



$f(x) = x^2$

$$\Delta = \frac{2-0}{n} = \frac{2}{n}$$

Left Endpoint

$$m_i = 0 + (i-1)\left(\frac{2}{n}\right) = \frac{2(i-1)}{n}$$

Right Endpoint

$$M_i = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$$

Using left endpoint

$$s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x = \sum_{i=1}^{n} f\left[\frac{2(i-1)}{n}\right]\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} \left[\frac{2(i-1)}{n}\right]^2\left(\frac{2}{n}\right)$$

$$= \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}$$

Using right endpoint

$$S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$$

$$= \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}$$

$$\boxed{s_n = \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2} < \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} = S(n)}$$

limit as $\Delta x \to 0$

Lower sum limit $\lim_{n\to\infty} s(n) = \lim_{n\to\infty}\left(\frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{8}{3}$

upper sum limit $\lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{8}{3}$

as $n \to \infty$, upper sum = lower sum

Limits of Upper and Lower sum

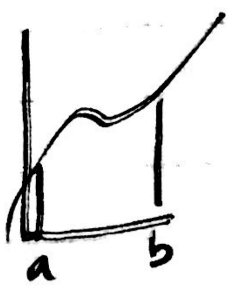
$f \to$ continuous, nonnegative $[a,b]$

$n \to \infty$

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\Delta x$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\Delta x$$
$$= \lim_{n\to\infty} S(n)$$

$\Delta x = \frac{b-a}{n}$ $f(m_i)$, $f(M_i)$ = minimum/max values of f

DEFINITION OF THE AREA OF A REGION

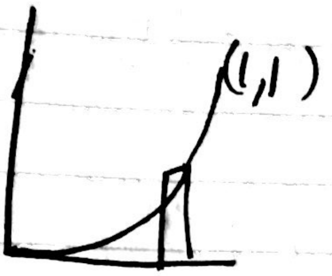


$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

$$x_{i-1} \leq c_i \leq x_i \qquad \Delta x = \frac{b-a}{n}$$

Example Finding Area by Limit Definition

$f(x) = x^3 \quad [0,1] \quad \Delta x = \frac{1}{n}$



$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \left[\frac{n^2(n+1)^2}{4}\right]$$

$$= \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right)$$

$$= \frac{1}{4}$$

[4.3] RIEMANN SUMS AND DEFINITE INTEGRALS

Definite Integrals definition

if f is defined on the closed interval [a,b] and the limit of Riemann sums over partitions Δ

$$\boxed{\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i}$$

exists, then f is integratable on [a,b] and the limit is denoted by

$$\boxed{\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,dx}$$

The limit is called the definite integral of f from a to b. The number a is the lower limit of integration, and the number b is the upper limit of integration

definite integral → number
indefinite integral → family of functions

CONTINUITY IMPLIES INTEGRABILITY

If a function f is continuous on the closed interval [a,b], then f is integrable on [a,b]. that $\int_a^b f(x)\,dx$ exists

Evaluating a definite integral as a Limit

$$\int_{-2}^{1} 2x \, dx \; , [-2,1] \quad \Delta x_i = \frac{b-a}{n} = \frac{3}{n}$$

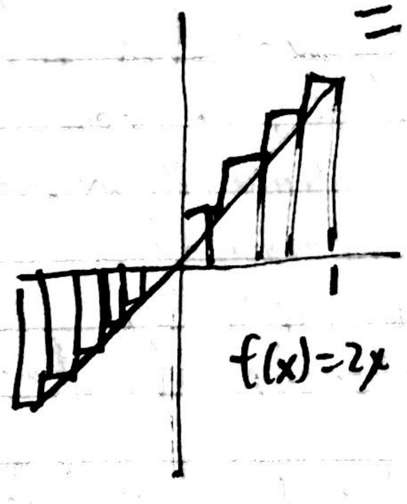
$c_i \Rightarrow$ right endpoint

$c_i = a + i(\Delta x) = -2 + \frac{3i}{n}$

$$\int_{-2}^{1} 2x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x$$

$$= \lim_{x \to \infty} \sum_{i=1}^{n} 2\left(-2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= -3$$



because definite integral is negative, it does not represent area of region.

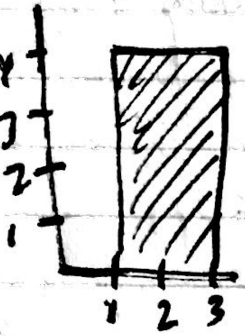
THE DEFINITE INTEGRAL AS THE AREA OF A REGION

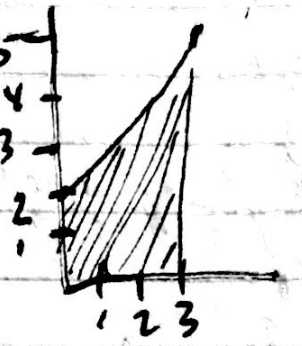
f ? continuous, nonnegative on $[a,b]$, area of region bounded by graph f, x axis, and vertical lines $x=a$ $x=b$ is given by

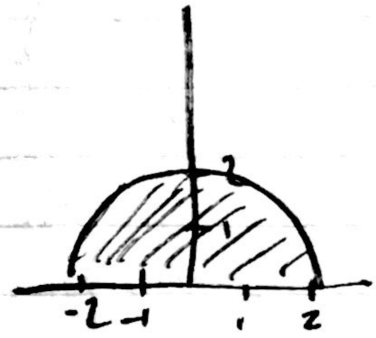
$$\text{Area} = \int_a^b f(x) \, dx$$

Areas of common Geometric Figures

a) $\int_1^3 4\,dx$ b) $\int_0^3 (x+2)\,dx$ c) $\int_{-2}^{2} \sqrt{4-x^2}\,dx$







$4(2)=8$ $\frac{1}{2}(3)(2+5)=\frac{21}{2}$ $\frac{1}{2}\pi(2^2)=2\pi$

Properties of Definite Integrals

1) $\int_a^a f(x)\,dx = 0$ 2) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

ADDITIVE INTERVAL PROPERTY

$$\int_a^b f(x)\,dx = \int_a f(x)\,dx + \int_c^b f(x)\,dx$$

ADDITIONAL PROPERTIES OF DEFINITE INTEGRALS

1) $\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$

2) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

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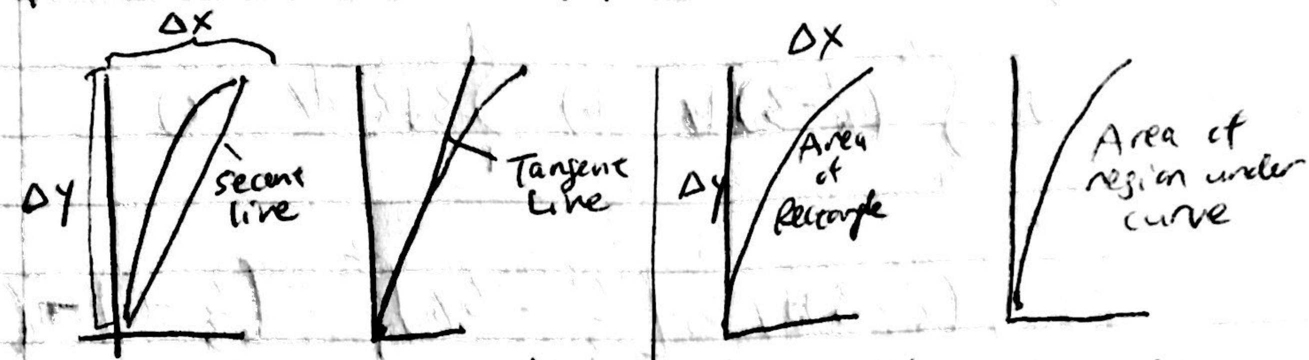
e.g. $\int_1^3 (-x^2 + 4x - 3)\,dx$

$= \int_1^3 (-x^2)\,dx + \int_1^3 4x\,dx + \int_1^3 (-3)\,dx$

$= \frac{4}{3}$

4.4 THE FUNDAMENTAL THEOREM OF CALCULUS



slope $= \frac{\Delta y}{\Delta x}$ | slope $\approx \frac{\Delta y}{\Delta x}$ | Area $= \Delta y \Delta x$ | Area $\approx \Delta y \Delta x$

Differentiation | Definite Integration

4.9 THE FUNDAMENTAL THEOREM OF CALCULUS

f is continuous on $[a,b]$

F is antiderivation of f on $[a,b]$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Guidelines for using the fundamental theorem of calculus

1. need to find antiderivation of f
2. $$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$
3. $$\int_a^b f(x)\,dx = \Big[F(x) + C\Big]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

not necessary to include constant integration C

eg.

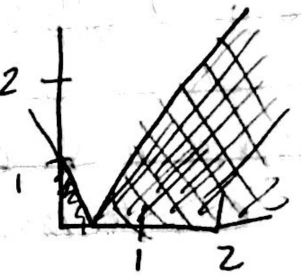
a) $\int_1^2 (x^2-3)\,dx$ b) $\int_1^4 3\sqrt{x}\,dx$ c)

a) $\int_1^2 (x^2-3)dx = \left[\frac{x^3}{3} - 3x\right]_1^2 = \left(\frac{8}{3}-6\right) - \left(\frac{1}{3}-3\right) = -\frac{2}{3}$

b) $\int_1^4 3\sqrt{x}\,dx = 3\int_1^4 x^{1/2}dx = 3\left[\frac{x^{3/2}}{3/2}\right]_1^4 = 2(4)^{3/2} - 2(1)^{3/2} = 14$

e.g. Definite Integral Involving Absolute Value

$\int_0^2 |2x-1|\,dx$



$$|2x-1| = \begin{cases} -(2x-1), & x<\frac{1}{2} \\ 2x-1 & x \geq \frac{1}{2} \end{cases}$$

$$\int_0^2 |2x-1|\,dx = \int_0^{1/2} -(2x-1)\,dx + \int_{1/2}^2 (2x-1)\,dx$$

$$= \frac{5}{2}$$

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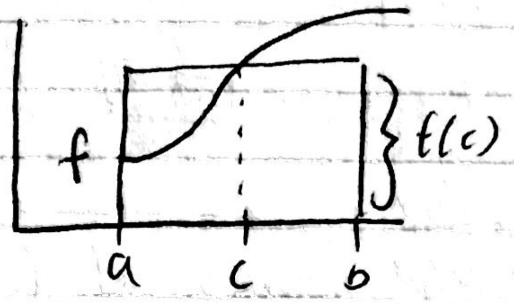
MEAN VALUE THEOREM FOR INTEGRALS

If f is continuous on closed interval $[a,b]$, then there exists a number c in $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)\,(b-a)$$

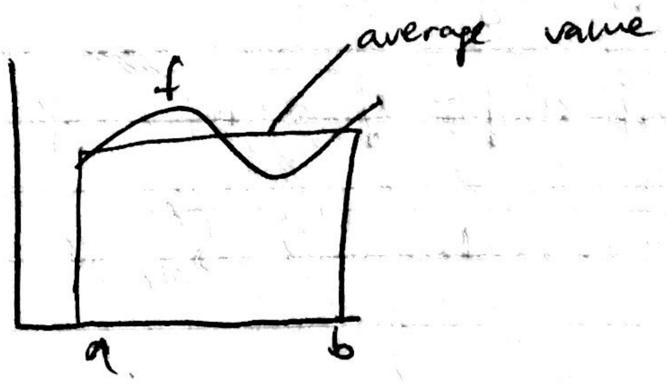


$f(c)$ = average value of f in $[a,b]$

AVERAGE VALUE OF A FUNCTION

If f is integral on closed interval $[a,b]$, then average value of f is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$



THE SECOND FUNDAMENTAL THEOREM OF CALCULUS

The Definite Integral as a Number

$$\int_a^{b \leftarrow \boxed{\text{constant}}} f(x)\, dx$$

a ← $\boxed{\text{constant}}$

$f(x)$ ← $\boxed{f \text{ is a function of } x}$

The Definite Integral as a Function of x

$$F(x) = \int_a^{x \leftarrow \boxed{F \text{ is a function of } x}} f(t)\, dx$$

a ← $\boxed{\text{constant}}$

$f(t)$ ← $\boxed{f \text{ is a function of } t}$

> **The Second Fundamental Theorem of Calculus**
>
> If f is continuous on an open interval I containing a, then, for every x in the interval
>
> $$\frac{d}{dx}\left[\int_a^x f(t)\, dt\right] = f(x)$$

e.g.

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\, dt\right]$$

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\, dt\right] = \sqrt{x^2+1}$$

because $f(t) = \sqrt{t^2+1}$ is continuous

eg Find derivative of $F(x) = \int_{\pi/2}^{x^3} \cos t\, dt$

$u = x^3$

$$F'(x) = \frac{dF}{du}\frac{du}{dx}$$

$$= \frac{d}{du}\left[\int_{\pi/2}^{x^3} \cos t\, dt\right]\frac{du}{dx}$$

$$= \frac{d}{du}\left[\int_{\pi/2}^{u} \cos t\, dt\right]\frac{du}{dx}$$

$$= \cos u\,(3x^2)$$

$$= \boxed{\cos x^3\,(3x^2)}$$

NET CHANGE THEOREM

The definite integral of rate of change of a quantity $F'(x)$ gives the total change, or net change, in that quantity on the interval $[a,b]$

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Solving a Particle Motion Problem

A particle is moving along a line so that its velocity is $v(t) = t^3 - 10t^2 + 29t - 20$ ft/s

a) What is the displacement of the particle on time interval $1 \leq t \leq 5$?

b) what is the total distance traveled by particle on time interval $1 \leq t \leq 5$?

a) displacement:

$$\int_1^5 v(t)\,dt = \int_1^5 (t^3 - 10t^2 + 29t - 20)\,dt$$

$$= \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_1^5$$

$$= \frac{25}{12} - \left(-\frac{103}{12}\right)$$

$$= \boxed{\frac{32}{3}\text{ ft}}$$

b) total distance: $\int_1^5 |v(t)|\,dt$

$$t^3 - 10t^2 + 29t - 20 = (t-1)(t-4)(t-5)$$

$[1,4] \Rightarrow v(t) \geq 0$

$$\int_1^5 |v(t)|\,dt = \int_1^4 v(t)\,dt - \int_4^5 v(t)\,dt \quad [4,5] \Rightarrow v(t) \leq 0$$

$$= \boxed{\frac{71}{6}\text{ ft}}$$

4.5 INTEGRATION BY SUBSTITUTION

ANTIDIFFERENTIATION OF A COMPOSITE FUNCTION

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,dx = F(g(x))+C$$

$$u=g(x) \quad du=g'(x)\,dx$$

$$\int f(u)\,du = F(u)+C$$

outside function

$$\int f(\underbrace{g(x)}_{\text{inside function}})\,\underbrace{g'(x)}_{\text{derivative of inside function}}\,dx = F(g(x))+C$$

e.g. Find $\int (x^2+1)^2(2x)\,dx$

$g(x)=x^2+1 \quad g'(x)=2x$

$f(g(x))=f(x^2+1)=(x^2+1)^2$

$$\int (x^2+1)^2(2x)\,dx = \frac{1}{3}(x^2+1)^3+C$$

e.g. $\int 5\cos 5x\,dx \qquad g(x)=5x \quad g'(x)=5$

$$\int (\cos 5x)\,5\,dx = \sin 5x + C$$

Constant Multiple Rule

$$\boxed{\int k\, f(x)\, dx = k \int f(x)\, dx}$$

you can multiple and divide by same number (constant)

e.g.

Find $\int x(x^2+1)^2\, dx$

$g(x) = x^2+1 \quad g'(x) = 2x$

$$\int x(x^2+1)^2\, dx = \int (x^2+1)^2 \left(\tfrac{1}{2}\right)(2x)\, dx$$

$$= \frac{1}{2} \int f(g(x))\, g'(x)\, dx$$

$$= \frac{1}{2}\left[\frac{(x^2+1)^3}{3}\right] + C$$

$$= \boxed{\frac{1}{6}(x^2+1)^3 + C}$$

CHANGE OF VARIABLES

if $u = g(x)$ $\quad du = g'(x)\,dx$

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du = F(u) + C$$

---

e.g. 1

Find $\int \sqrt{2x-1}\,dx$ $\qquad u = 2x-1 \quad du = 2\,dx$

$\qquad \sqrt{u} = \sqrt{2x-1} \quad dx = \frac{du}{2}$

$$\int \sqrt{2x-1}\,dx = \int \sqrt{u}\left(\frac{du}{2}\right)$$

$$= \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C$$

$$= \frac{1}{3}(2x-1)^{3/2} + C$$

---

e.g. 2

$\int x\sqrt{2x-1}\,dx$ $\qquad u = 2x-1 \qquad dx = \frac{du}{2}$

$\qquad u+1 = 2x$

$\qquad x = \frac{u+1}{2}$

$$\int x\sqrt{2x-1}\,dx = \int \left(\frac{u+1}{2}\right) u^{1/2}\left(\frac{du}{2}\right)$$

$$= \frac{1}{4}\int \left(u^{3/2} + u^{1/2}\right) du$$

$$= \frac{1}{4}\left(\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C = \boxed{\frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C}$$

GUIDELINES FOR MAKING A CHANGE OF VARIABLES

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the inner part of a composite function.
2. Compute $du = g'(x)\,dx$
3. Rewrite the integral in terms of variable u
4. Find the resulting integral in terms of u
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

The General Power Rule for Integration

$$\int [g(x)]^n g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C \quad n \neq -1$$

So...

$$\int u^n\,du = \frac{u^{(n+1)}}{n+1} + C \quad n \neq -1$$

CHANGE OF VARIABLES FOR DEFINITE INTEGRALS

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

e.g. Evaluate $\int_0^1 x(x^2+1)^3\,dx$

$u = x^2+1$
$du = 2x\,dx$

$$= \frac{1}{2}\int_0^1 (x^2+1)^3 (2x)\,dx$$

$$= \frac{1}{2}\int_1^2 u^3\,du = \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 = \boxed{\frac{15}{8}}$$

e.g. 2

$$A = \int_1^5 \frac{x}{\sqrt{2x-1}}\,dx \qquad u = \sqrt{2x-1}$$

$$u^2 = 2x - 1$$

$$x = \frac{u^2+1}{2}$$

$$u\,du = dx$$

Lower Limit

$x = 1$

$u = \sqrt{2-1} = 1$

Upper limit

$x = 5$

$u = \sqrt{10-1} = 3$

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \int_1^3 \frac{u^2+1}{2u}\,u\,du$$

$$= \frac{1}{2}\int_1^3 (u^2+1)\,du$$

$$= \frac{1}{2}\left[\frac{u^3}{3} + u\right]_1^3$$

$$= \frac{1}{2}\left(9 + 3 - \frac{1}{3} - 1\right)$$

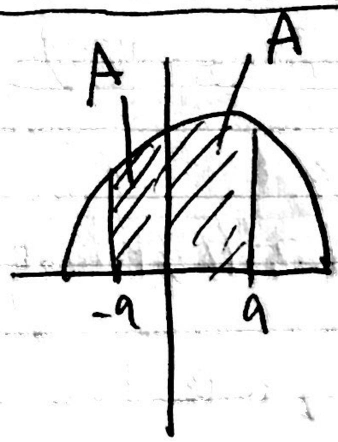
$$= \frac{16}{3}$$

INTEGRATION OF EVEN AND ODD FUNCTIONS

let f be integrable on the closed interval $[-a, a]$

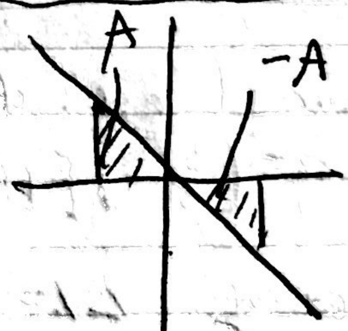
1. If f is an even function then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

2. If f is an odd function then $\int_{-a}^{a} f(x)dx = 0$



total A = 2A

even



total A = A − A = 0

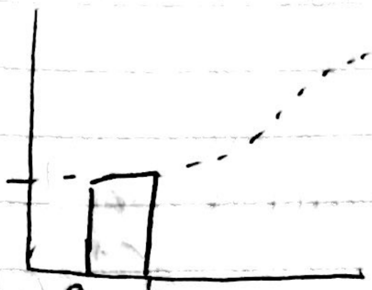
odd

4.6 NUMERICAL INTEGRATION

The Trapezoidal Rule

area of trapezoid $= h\left(\frac{b_1+b_2}{2}\right)$

$$= \left(\frac{b-a}{n}\right)\left[\frac{f(x_0)+f(x_1)}{2}\right]$$



Sum of areas of n trapezoids

$$\text{Area} = \left(\frac{b-a}{2n}\right)[f(x_0)+2f(x_1)+2f(x_2)+\dots+f(x_n)]$$

Letting $\Delta x = (b-a)/n$, take limit $n \to \infty$

$$\lim_{n\to\infty}\left(\frac{b-a}{2n}\right)[f(x_0)+2f(x_1)+\dots 2f(x_{n-1})+f(x_n)]$$

$$= \lim_{n\to\infty}\left[\frac{[f(a)-f(b)]\Delta x}{2} + \sum_{i=1}^{n} f(x_i)\Delta x\right]$$

$$= \lim_{n\to\infty}\frac{[f(a)-f(b)](b-a)}{2n} + \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= 0 + \int_a^b f(x)\,dx$$

THE TRAPEZOIDAL RULE

Let f be continuous on $[a,b]$. The Trapezoidal Rule for approximating $\int_a^b f(x)\,dx$ is given by

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\dots f(x_n)\right]$$

Moreover as $n \to \infty$, right hand sum approaches $\int_a^b f(x)\,dx$

e.g.

Use trapezoidal Rule to approximate

$$\int_0^\pi \sin x\,dx$$

$$-\cos x\Big]_0^\pi = \begin{array}{l}-\cos\pi - \\ (-\cos 0) \\ 1+1=\boxed{2}\end{array}$$

$n=4 \qquad \Delta=\frac{\pi}{4}$

$$\int_0^\pi \sin x\,dx \approx \frac{\pi}{8}\left(\sin 0 + 2\sin\frac{\pi}{4} + 2\sin\frac{\pi}{2} + 2\sin\frac{3\pi}{4} + \sin\pi\right)$$

$$= \frac{\pi}{8}\left(0+\sqrt{2}+2+\sqrt{2}+0\right) = \frac{\pi(1+\sqrt{2})}{4} \approx 1.896$$

$n=8 \qquad \Delta x=\frac{\pi}{8}$

$$\int_0^\pi \sin x\,dx \approx \frac{\pi}{16}\left(\sin 0 + 2\sin\frac{\pi}{8} + 2\sin\frac{\pi}{4} + 2\sin\frac{3\pi}{8} + 2\sin\frac{\pi}{2} + 2\sin\frac{5\pi}{8} + 2\sin\frac{3\pi}{4} + 2\sin\frac{7\pi}{8} + \sin\pi\right)$$

$$\approx 1.974$$

RIEMANN SUMS

$f(x) = \sqrt{x} \qquad 0 \le x \le 1$

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\,\Delta x \qquad \Delta x_i = \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2}$$

$$= \frac{2i-1}{n^2}$$

limit:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\frac{i^2}{n^2}} \left(\frac{2i-1}{n^2} \right)$$

$$= \lim_{n \to \infty} \frac{1}{n^3} \sum_{i=1}^{n} (2i^2 - i)$$

$$= \lim_{n \to \infty} \frac{4n^3 + 3n^2 - n}{6n^3}$$

$$= \frac{2}{3}$$

Definition of Riemann Sum

let f be defined on the closed interval $[a,b]$, and let Δ be a partition of $[a,b]$ given by

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval. If c_i is <u>any</u> point in the ith subinterval $[x_{i-1}, x_i]$, then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x, \qquad x_{i-1} \le c_i \le x_i$$

is called a Ri[illegible] Δ